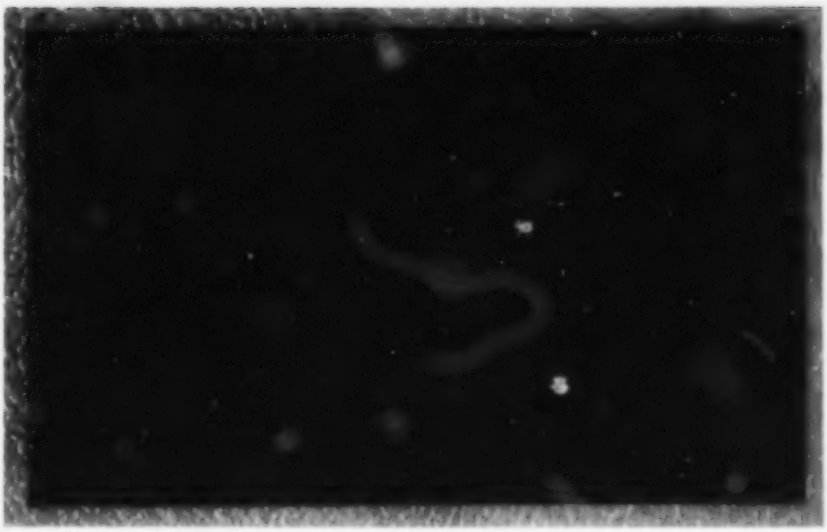
UNIVERSITÉ DE MONTRÉAL



**CENTRE DE RECHERCHE SUR LES TRANSPORTS**

CASE POSTALE 6128, SUCCURSALE A, MONTRÉAL, QUÉBEC, H3C 3J7

CRT-618

# A UNIFIED THEORY OF AGGREGATION: SIMILARITY AND SEPARABILITY RECONSIDERED

by

Nicole M. Fortin*

* Département de Sciences économiques and Centre de recherche sur les transports, Université de Montréal.

Most of the research for this paper was conducted as part of the author's Ph.D. dissertation at the Department of Economics, University of British Columbia. Latter research was supported by CAFIR grant #114-81, Université de Montréal and by the MESST, Québec. The author wishes to thank Charles Blackorby, John Cragg, Erwin Diewert, Terence Gorman, William Schworm, Thomas Stoker and John Walsh for useful comments.

Université de Montréal

Centre de recherche sur les transports - Publication #618
Département de Sciences économiques - Cahier #8908

March 1989

*Abstract*

Aggregation problems in macro models, especially in representative agent models, arise from the use of aggregate variables that do not capture all the distributional properties of the micro relations. This paper presents a stochastic framework to analyze and evaluate aggregation biases that measure the divergence between a macro relation and the corresponding "true" aggregate relation. The analysis of potential sources of biases shows that the functional form restrictions of exact-aggregation models (Gorman, 1968, Blackorby and Schworm, 1984, 1988) are neither necessary nor sufficient conditions for consistent aggregation, if the aggregates are taken to be the usual totals or averages. Traditionally, a high degree of similarity among agents was required to ensure exact aggregation, as in the Gorman polar form. Here, economic diversity provides an alternative set of circumstances under which aggregation biases may be minimized. Distributional conditions can be seen as substitutes for functional separability. The relationships between exact-aggregation macro parameters and their microfoundations are studied. Theil's (1954, 1971) statistical results concerning the relations between micro and macro parameters are formalized at the population level and generalized to non-linear functions. Conditions under which exact-aggregation macro parameters possess stable microdefinitions are obtained. They explain the results of Fisher's (1971) simulation experiments investigating the empirical accuracy of the macro Cobb-Douglas production function. An empirical illustration of the theoretical results is provided.

KEYWORDS: Aggregation of economic relations, aggregation biases, representative agent models, separability, Gorman polar form, distributional restrictions, Cobb-Douglas production function.

*Résumé*

Les problèmes d'agrégation dans les modèles macroéconomiques, spécialement dans les modèles de type agent représentatif, découlent de l'utilisation de variables agrégées qui ne capturent pas toutes les propriétés distributionnelles des relations micros. Ce papier présente un cadre conceptuel stochastique pour analyser et évaluer les biais d'agrégation, qui mesurent la divergence entre la relation macro et la relation agrégée "vraie" correspondante. L'analyse des sources potentielles de biais démontre que les restrictions sur les formes fonctionnelles dictées par les modèles d'agrégation exacte (Gorman, 1968, Blackorby and Schworm, 1984, 1988) ne sont ni nécessaires ni suffisantes pour assurer une agrégation consistente, si les agrégats utilisés sont des totaux ou des moyennes. Traditionnellement, un haut degré de similarité entre agents —tel que requis par la forme polaire de Gorman— était nécessaire pour obtenir l'agrégation exacte. Ici, un autre ensemble de circonstances sous lesquelles les biais d'agrégation peuvent être minimisés provient de la diversité économique. Des conditions distributionnelles peuvent se substituer aux conditions de séparabilité sur les formes fonctionnelles. Les relations entre les paramètres macros d'agrégation exacte et leurs microfondements sont étudiées. Les résultats statistiques de Theil (1954, 1971) concernant les relations entre les paramètres micros et macros sont formalisés au niveau de la population et généralisés aux fonctions non linéaires. Les conditions sous lesquelles les paramètres macros d'agrégation exacte possèdent des définitions micros stables sont dérivées. Elles expliquent les résultats des simulations de Fisher (1971), qui cherchaient à expliquer la robutesse empirique de la fonction de production Cobb-Douglas au niveau macro. Une illustration empirique des résultats théoriques est incluse.

MOTS-CLÉS: Agrégation des relations économiques, biais d'agrégation, modèles de type agent représentatif, séparabilité, forme polaire de Gorman, restrictions distributionnelles, fonction de production Cobb-Douglas.

# 1. INTRODUCTION

Aggregation problems arise when the workings of the economic system are described at the macro level, whether the macro description mimics individual behavior (the representative agent view), or whether it is seen solely as a relation between aggregates (the view of Keynes and others). [2]

With the first approach, consistency (or exactness) in aggregation is a logical imperative. [3] Jevons (1879), who first explicitly introduced the representative agent view, had recognized the possibility of discrepancies between macro and micro laws, discrepancies that could be bridged only under special distributional circumstances. [4] Representative agent models are now common place in the analysis of real business cycles (Kydland and Prescott, 1982), in asset pricing models (Lucas, 1978, Cox, Ingersoll and Ross, 1985), in rational expectations models based on the Lucas supply function and in other macro models. Yet, they are often based on forms that cannot be aggregated exactly—such as the Cobb-Douglas production function—without

[2] As argued by Friedman (1976, p.316-317). Nevertheless, microfoundations for Keynesian aggregate supply have been proposed (Casarosa, 1981).

[3] The representative agent view assumes the existence of an agent whose behavior, if replicated $n$ times, reproduce the behavior of the market, the community or, more generally, of the aggregate unit. The behavior of the replica must correspond exactly to the aggregative behavior of $n$ representative individual.

[4] "The general forms of the laws of Economics are the same in the case of individuals and nations; and, in reality, it is a law operating in the case of multitudes of individuals which gives rise to the aggregate represented in the transactions of a nation. [...] It would not of necessity, happen that the law would be exactly the same in the case of aggregates and individuals, unless all those individuals were of the same character and position as regards wealth and habits; but there would be a more or less regular law to which the same kind of formulæ would apply. " (Jevons, 1879, p. 16-17)

references to the implications of such inconstitenties. [5]

With the second approach, a consistent link between macro and micro relations is needed if the macro model is used to predict the aggregate outcomes of policy changes or market shocks that are transmitted via the micro relationships. In his *Grundsätze* (1871), Menger had hinted already at possible (aggregation) errors if the functional relations implied by mere quantitative aggregates, such as "national wealth", were used for policy analysis:

> But when inferences running from the magnitude of the national wealth to the welfare of a people, or when a phenomena resulting from contacts between the various economizing individuals, are involved, the concept of national wealth in the literal sense of the term must necessarily lead to frequent errors. (Menger, 1871/1950, p.113)

In models of multi-sectoral economies, policy recommendations based on macro models without solid microfoundations have been shown to be misleading (Blinder and Mankiw, 1984, Biser and Sichel, 1988). Yet, when such models are used—in the description of investment behavior, for example—, nothing can be said about the seriousness of these errors.

In both cases, traditional exact-aggregation theories (Nataf, 1948; Gorman, 1953, 1968; Fisher, 1965, 1969, 1982; Blackorby and Schworm, 1984, 1988) have offered little help to macro modelers, as they impose stringent conditions that are often impossible to satisfy empirically. This paper attempts to correct this unsatisfactory situation by proposing a framework within which the sources and

[5] The Cobb-Douglas is generally not exact in aggregation in a representative agent framework. However, it can be seen as the outcome of other aggregative schemes (Houthakker, 1955, Chetty and Heckman, 1986).

causes of aggregation biases can be identified. The potential aggregation problems then could be corrected either by modifying the choice of functional form or by including the statistics needed to complete the aggregation structure.

Developped in the context of producer theory, the framework can be extended to any economic relations that are described parametrically. It is based on a probabilistic interpretation of the production characteristics of a population of firms, which encompasses that of previous stochastic aggregation models. These models have considered either the technological coefficients (Houthakker, 1955, Johansen, 1972, Sato, 1975, Hildenbrand, 1981) or the variables (Stoker, 1984, Lewbel, 1986a, b) to be stochastic. Here, the technological parameters, the factor inputs (and the prices, when applicable) are taken to be different for each firm and assumed to belong to a joint probability distribution. "True" aggregate relations then capture all the distributional properties of the micro relations. Theil's (1954, 1971) statistical results concerning the relations between micro and macro parameters are formalized at the population level and generalized to non-linear functional forms.

A decomposition of "true" aggregate functions in terms of moments of the characteristics' distribution enables both the functional form restrictions of exact-aggregation models and the distributional constraints of stochastic aggregation models to be considered. This strategy makes the approach "distribution free" in the sense that it applies identically to any unknown distribution, as long as the appropriate moments exist. Changes in the population, both at the intensive and extensive margins, may entail dramatic changes in the moments of the distribution of characteristics and thus are accounted for implicitly.

An exact-aggregation requirement is stated in terms of the equality of the out-

come of a given macro relation and of the one resulting from the "true" aggregation of the micro relations. Discrepancies result in aggregation biases. The representative agent's problem is considered next as a special case of a completely specified macro function. In the case of an *average-representative* firm, an output aggregation bias is explicitly derived, exactly for polynomials and approximately for non-polynomial functional forms. When the macro function is not completely specified —for example, when its parameters are left to be estimated—, the exact-aggregation macro parameters that retain the same microdefinitions over time, will be constant only under certain conditions. These conditions explain the results of Fisher's (1971) simulation experiments investigating the empirical accuracy of the macro Cobb-Douglas production function. Deviations from time-invariant microfoundations are used to evaluate time-biases in the exact-aggregation macro parameters. [6]

In this stochastic framework, previous exact-aggregation conditions (Gorman, 1968, Blackorby and Schworm, 1984, 1988) are found to be no longer necessary nor sufficient to ensure exact-aggregation, using traditional totals and averages as aggregates. Commonality across firms of specific components of the profit functions and separability in the capital inputs are shown to be too strong restrictions. A proposed weaker zero-covariances-sum condition between appropriate components of the profit functions extends the linear independence condition of Lau's (1982) fundamental theorem of exact-aggregation. Before, a high degree of similarity among firms was thought to be an essential condition of exact-aggregation. Here,

[6] Biases in macro parameters are time-dependent, whereas biases in outcomes are cross-sectional. A statistical notion of aggregation bias in the macro parameters was introduced by Theil (1954, p.122ff.; 1971, p.560) in the context of linear aggregation; it also was discussed in Allen (1959, p.700), Van Daal (1980) and Van Daal and Merkies (1984, p.184ff.).

economic diversity, when it entails the annihilation of the sum of higher and central moments of the components of the micro functions, emerges as an alternative set of circumstances under which aggregation biases may be minimized.

The paper is organized as follows. Section 2 presents the basic framework in terms of the aggregation of production functions outside of equilibrium (Klein's view, 1946), which illustrates the aggregation of ad hoc relations. Section 3 extends the model to the aggregation of profit functions, examines the implications of perfect competition on aggregation biases, and reevaluates the aggregative properties of Gorman type forms. Section 4 reports the empirical evidence substantiating the theoretical results, using sectoral data from Canadian input-output tables. Section 5 concludes with a discussion of the main results.

## 2. BASIC FRAMEWORK

### *2.1. Definitions and Decomposition of the Aggregates*

The aggregation of firm-specific production function, $y^f = g^f(x^f)$ $(f = 1,2, \ldots, F)$, where $y^f \in \mathbb{R}_+$ denotes the level of output and $x^f \in \mathbb{R}_+^N$ is a vector of inputs, is considered first as an example of the aggregation of arbitrary micro functions not constrained by an optimization process. [7] All micro functions need to belong to the same class of functional form using a finite parametrization. This is not unduly restrictive. If the technologies $g^f$ are known, if their number $F$ is finite and if they can be described using a finite number of parameters $S^f$, then it is always possible to represent such production technologies using the same parametric functional form, by including the technological parameters as arguments of the

[7] The aggregation of optimized production processes is easier to handle by considering the aggregation of profit functions which is taken on in the next section.

function: $y^f = g(x^f, \theta^f) \equiv g(z^f)$, where $z^f$ are called production characteristics, and where $g$ is called the augmented production function and is identical for all firms. [8] If the $g^f$ are unknown, the choice of the same flexible functional form for all firms is concurrent with current econometric practice. [9]

The production characteristics $(z^1, \ldots, z^F) \in \mathbb{R}^F \times \mathbb{R}^{N+S}$ are interpreted as realizations of a random vector $Z \in \mathbb{R}^{N+S}$. This interpretation can be seen as an ex post mathematical representation of the distribution of factor inputs and technological parameters. The random vector $Z$ is distributed according to a joint density $\phi_Z(z)$ of inputs and parameters. The first, other higher and central moments of the density $\phi_Z(z)$ are assumed to exist. [10] The first order moment, or mean, is defined as

$$\mu_z \equiv \mathbf{E}(Z) \equiv \int_{\Omega_z} z\, \phi_Z(z)\, dz, \tag{2.1}$$

where $\Omega_z$ is the domain of definition of $Z$ and where the integral of a vector is defined as a vector of integrals, so that $\mu_z = (\mu_{x_1}, \ldots, \mu_{x_N}, \mu_{\theta_1}, \ldots, \mu_{\theta_S}) \in \mathbb{R}^{N+S}$. The aggregates of the micro characteristics that will enter the macro function are defined in terms of functions of the distribution of $Z$ and are denoted by $\psi_z(Z)$. Similarly, "true" output aggregates, in the sense of capturing all the distributional properties

[8] For example, the worst case would take the sum of firm-specific production functions to be the common function. Let $\delta = (\delta^1, \ldots, \delta^F)$, let $\alpha = (\alpha_1^1, \ldots, \alpha_{S^1}^1, \ldots, \alpha_1^F, \ldots, \alpha_{S^F}^F)$, where $\alpha_1^f, \ldots, \alpha_{S^f}^f$ are the parameters of $g^f$ and $S^f$ is the number of parameters of $g^f$. Write $g^f(x^f) \equiv \tilde{g}(x^f, \alpha^f)$ in augmented form. Let $g(\xi, \delta, \alpha) \equiv \sum_{f=1}^F \delta^f\, \tilde{g}(\xi, \alpha^f)$, $g: \mathbb{R}^{F+N+S} \to \mathbb{R}$, where $\xi$ is a vector of dimension $N$ and $S = \sum_{f=1}^F S^f$. Then $g(x^f, \theta^f) = g^f(x^f)$ for $\theta^f \equiv (d^f, a^f) \in \mathbb{R}^{F+S}$ where $d^f = (0, \ldots, 1^f, \ldots, 0)$ and $a^f = (0, \ldots, \alpha_1^f, \ldots, \alpha_{S^f}^f, \ldots, 0)$. Note that $g$ represents a specific function and not a class of admissible functions.

[9] Then $S^f \equiv S$, for all $f$, where $S$ is the dimension of the vector of parameters $\theta^f$.

[10] The functional form of $g$ actually determines which precise central and non-central higher moments are needed in each case.

of the micro relations, are defined, provided that the integral is well-defined, as:

$$(2.2) \qquad \mathcal{Y} \equiv F \cdot \mathbf{E}[g(Z)] \equiv F \cdot \int_{\Omega_z} g(z)\, \phi_Z(z)\, dz \quad \text{or} \quad \mu_y \equiv \frac{\mathcal{Y}}{F}$$

where $g: \mathbb{R}^{N+S} \rightarrow \mathbb{R}$, so that $\mathcal{Y} \in \mathbb{R}$. In terms of observed values, these aggregates are the usual statistics, the totals and averages, used in non-stochastic exact-aggregation models, such as Fisher's (1965, 1969, 1982). This approach presumes that products are homogeneous (non-differentiated); in cases of aggregation across an economy, constant dollars can serve as the common non-differentiated unit of account.

Stochastic aggregation models have utilized expressions such as expected output (2.2) and have attempted to solve them for specific $g(\cdot)$ and $\phi_Z(\cdot)$. In the Houthakker-Johansen-Sato model, if the density of capacity output follows a Pareto distribution, and if all firms operate with fixed (or variable) factor input proportions, then the aggregate production function will be Cobb-Douglas (or CES). In more recent models (Stoker, 1984 and Lewbel, 1986a, b), the distribution is assumed to be *mean full dimensional*, that is, parametrizable in terms of the mean of the variables. In Stoker (1986), the parametrization is such that an invertible mapping between the parameters and the mean of the variables exists. [11] With these modeling strategies, the aggregation problem is partially transfered onto the parametric space of the characteristics' distribution and requires non-trivial distributional assumptions. [12] Also, these assumptions are valid often only when coupled with specific micro

[11] Let $\delta$ parametrize the density $\phi_Z(z) \equiv \phi_Z(z|\delta)$, and let $\mu_z = \mathbf{E}(Z|\delta) = H(\delta)$. *Mean full dimensionality* assumes that there exists $\delta$ such that $H(\delta) \in \mathbb{R}^{N+S}$. Further, Stoker (1986, p.174) assumes that $\delta = H^{-1}(\mu_z)$ exists, implying that $\mu_y = \int g(z)\phi_Z(z|\delta)\, dz = G^*(\delta) = \tilde{G}(H^{-1}(\mu_z)) = G(\mu_z)$. The latter is equivalent (or even stronger than) to an exact-aggregation requirement, unless $z$ is allowed to include powers or cross-products of the variables.

[12] In Lewbel's (1986b) model, log-log micro relations can be aggregated exactly

functions.

This paper proposes an alternative approach that is, to write aggregate output in terms of first and other moments of the characteristics that can be evaluated empirically and that have an interesting intuitive appeal. If the augmented production function $g(z^J)$ is a polynomial, it is possible to express its expectation $\mathbf{E}[g(Z)]$ in terms of first and higher central moments of the characteristics. The former are used to define the characteristics' aggregates. The latter provide a measure of the interdependence of the characteristics, which is akin to a population measure of "separability". If the characteristics are distributed independently, covariance terms vanish.

Assuming for simplification that the random vector $Z \in \mathbb{R}^L$, let $g(Z)$ be L-linear: $g(Z) = \sum_{i_1,i_2,\ldots,i_L=0}^{1} Z_1^{i_1} Z_2^{i_2} \cdots Z_L^{i_L}$, where some components of $Z$ may be technological parameters, then

$$(2.3) \qquad \mathbf{E}[g(Z)] = \sum_{i_1,i_2,\ldots,i_L=0}^{1} \mathbf{E}[Z_1^{i_1} Z_2^{i_2} \cdots Z_L^{i_L}] = \sum_{i_1,i_2,\ldots,i_L=0}^{1} \mu[i_1, i_2, \ldots, i_L],$$

where $\mathbf{E}[Z_1^{i_1} Z_2^{i_2} \cdots Z_L^{i_L}] \equiv \mu[i_1, i_2, \ldots, i_L]$ is a moment of order $\eta = i_1 + i_2 + \cdots + i_L$ of the random vector $(Z_1, Z_2, \ldots, Z_L)$. Similarly, let the central moments be denoted by $m[i_1, i_2, \ldots, i_L] \equiv \mathbf{E}[(Z_1 - \mu[1,0,\ldots,0])^{i_1} (Z_2 - \mu[0,1,0,\ldots,0])^{i_2} \cdots (Z_L - \mu[0,0,\ldots,1])^{i_L}]$. Let $e_k = (0,\ldots,1_k,\ldots,0)$, $k = 1,2,\ldots,L$, be the canonical basis

---

into a log-log macro relation under the condition that the distribution of the explanatory variables be *mean scaled*, that is, homogeneous of degree 0 in $z$ and $\mu_z$. However, in many instances where distributional data exists (such as personal income and manufacturing sales), Lewbel found that the empirical evidence did not support this requirement.

of $\mathbb{R}^L$. Any moment ($\eta \leq L$) of $Z \in \mathbb{R}^L$ can be expressed as:

$$\begin{aligned}\mu[i_1, i_2, \ldots, i_L] &\equiv \mu[\sum_{k=1}^{L} i_k e_k] \\ &= \prod_{k=1}^{L} \mu[i_k e_k] + \sum_{j_1 j_2 \cdots j_L = 0}^{1} m[\sum_{k=1}^{L} j_k e_k]\ \mu[\sum_{k=1}^{L} (i_k - j_k) e_k],\end{aligned} \tag{2.4}$$

where $\mu[0, 0, \ldots, 0] = 1$, $\mu[\sum_{k=1}^{L} j_k e_k] = 0$ if $\sum_{k=1}^{L} j_k e_k < 0_L$ and $m[\sum_{k=1}^{L} j_k e_k] = 0$ if $\sum_{k=1}^{L} j_k e_k \leq 1_L$. [13]

The expectation of $g(Z)$ becomes

$$\begin{aligned}\mathbf{E}[g(Z)] &= g[\mathbf{E}(Z)] + \sum_{i_1 i_2 \cdots i_L = 0}^{1} \Big( \sum_{j_1 j_2 \cdots j_L = 0}^{1} m[\sum_{k=1}^{L} j_k e_k]\ \mu[\sum_{k=1}^{L} (i_k - j_k) e_k] \Big) \\ &= g[\mathbf{E}(Z)] + \mathbf{M}[g(Z)].\end{aligned} \tag{2.5}$$

The right hand side terms, excluding $g[\mathbf{E}(Z)]$, are called the sum of higher moments of expected output $\mathbf{M}[g(Z)]$. [14]

Decomposition (2.5) can be extended to any polynomial, writing the $j$-th powers of a variable as a $j$-product, and applied to truncated series expansions. [15] Despite the limits of its applicability, the approach is still rewarding since decompositions in terms of product of functions of the characteristics are useful as will be shown in Section 3.2.

### *2.2. Exact-Aggregation Requirement and Resulting Biases*

[13] For example, if $z \in \mathbb{R}^3$, $\mathbf{E}[Z_1 Z_2 Z_3] \equiv \mu_{111} = \mu_{100}\mu_{010}\mu_{001} + \mu_{100} m_{011} + \mu_{001} m_{101} + \mu_{010} m_{110} + m_{111} = \mathbf{E}(Z_1)\mathbf{E}(Z_2)\mathbf{E}(Z_3) + \mathbf{E}(Z_1) m_{011} + \mathbf{E}(Z_2) m_{101} + \mathbf{E}(Z_3) m_{110} + m_{111}$.

[14] Note that despite the similarly of expression (2.5) with Jensen's (1906) inequality on convex functions, the latter does not apply directly. The usual concavity in factor inputs of the production function applies to the first $N$ arguments of $g$—the factor inputs— but does not extend to the $S$ others—the technological parameters. Because the augmented production function $g$ includes the technological parameters of the production function $g^f$, its global curvature is undetermined.

[15] Alternatively, moments decomposition of non-polynomial functional forms, that are continously differentiable, can be approximated by decomposing the expectation of the functions' Taylor series expansion about the mean.

The exact-aggregation requirement is defined similarly to that of other exact-aggregation models by the equality:

$$Y \equiv F \cdot \mu_y = G(\psi_z(Z)), \tag{2.6}$$

where $\psi_z(Z) \in \mathbb{R}^{N+S}$ is an aggregate of the micro characteristics and $G: \mathbb{R}^{N+S} \rightarrow \mathbb{R}$.

If a specific macro function $G$ is assumed *a priori*, the exact-aggregation requirement (2.6) serves to test the correspondence between this macro relation and its microfoundations. If the form of the macro function is not known, which is generally the case, it may be assumed that the augmented macro function can be described in the same way as the augmented micro functions are: $G \equiv g$. [16] This assumption is more general then the representative agent view of macroeconomic modeling, since the behavioral parameters can differ across agents and at the macro level. [17]

If the exact-aggregation requirement (2.6) fails to be satisfied, a total output aggregation bias can be defined as:

$$B_y = F \cdot \int_{\Omega_z} g(z)\, \phi_Z(z)\, dz - G(\psi_z(Z)). \tag{2.7}$$

Note that the augmented macro function may be fully specified, as in the *average-representative* firm approach below, or may not, as in most exact-aggregation models

[16] Any micro technology, represented by the augmented production function $g$, is interpreted as a particular draw from the population of technological parameters. This assumption supposes that the choice of macro technology is simply another draw and should not be restrictive.

[17] In macroeconomic modeling (Lucas and Rapping, 1969; Sargent, 1978), the representative agent conveniently models the behavior of a typical agent which, if replicated $n$ times, mimics aggregate behavior. Identical parameters for all agents are assumed.

where the macro parameters are unknown. The latter may reflect the viewpoint that macro parameters cannot a priori be derived from the distribution of micro characteristics. Exact aggregation defines the set of such possible relations.

### *2.3. The* Average-Representative *Firm : a Completely Specified Case*

The representative agent's problem considered here asks, whether there exists a representative agent whose behavior can be derived from the micro behavior of non-identical agents. The *average-representative* firm's technology, which is also the macro technology, is thought of as an average of all firms technologies: $G \equiv g$, and $\psi_z(Z) = (\mu_x, \mu_y)$. It corresponds to some extent to the Marshallian concept of a representative firm: "a representative firm is in a sense an average firm" (Marshall, 1920, p.318).

The exact-aggregation requirement and the aggregation bias are stated more adequately in terms of averages: $\mu_y = g(\mu_z)$ implying $B_{\mu_y} = \mu_y - g(\mu_z)$. If $g$ is a polynomial, the average output aggregation bias is given by the higher moments of expected output: $B_{\mu_y} = \mathrm{M}[g(Z)]$. If $g$ possesses a non-polynomial functional form but is continuously differentiable, Keller (1980) has suggested the use of an approximation derived from the Taylor's series expansion: $B_{\mu_y} \approx \frac{1}{2}\mathrm{tr}(\bar{H}C)$, where tr is trace operator, $\bar{H}$ the Hessian matrix evaluated at $\mu_z$ and $C$ the covariance matrix. [18] Thus, unless the micro relations are such as to entail the disappearance of the sum of higher moments terms, it is not possible to find an *average-representative* firm that possesses exact microfoundations.

**Proposition 1.** *The aggregation bias of the average-representative firm will be*

[18] Given the skewness of presumed distribution of firms across the economy, a better approximation would include third order terms.

*null if and only if the sum of higher moments ($\eta > 1$) of the characteristics in the average output aggregation $B_{\mu_y}$ bias vanish.*

*Proof:* Follows from the moments' decomposition (in exact or series' form) of the average output aggregation bias. See appendix. ◇

This condition may be more or less restrictive than requiring that the micro relations be linear. Since linearity in factor inputs $x^f$ is generally equivalent to 2-linearity in the production characteristics $z^f$, it does not guarantee the elimination of the sum of covariance terms. On the other hand, the elimination of higher order terms may result from properties of the distribution of $Z$ allowing a non-linear production function.

### *2.4. Microfoundations of Exact-Aggregation Macro Parameters*

By contrast with the *average-representative* firm approach, exact-aggregation models generally specify aggregates for the factor inputs only. For example, the macro factor inputs may be taken to be $\psi_x(X) = F \cdot \mu_x \equiv \mathcal{X}$, evaluated as the sum of the inputs. If the macro parameters are left unspecified, exact aggregation will require that the macro parameters $\theta^M$ belong to the zero-bias $S$-manifold described by the following equation:

$$G(\mathcal{X}, \theta^M) - \mathcal{Y} = 0. \tag{2.8}$$

As pointed out by Richmond (1976) in a slightly different context, the aggregation problem can seen as an identification problem. The choice of a particular vector from this manifold may be helped by imposing the restriction that the exact-aggregation macro parameters be identical over time. This restriction may be thought as being

ad hoc, but is common enough to justify its investigation. [19]

If the macro parameters are assumed constant over time, the relevant aggregation issue is whether such macro parameters possess stable microfoundations. Time-subscripts and indexes are now introduced. If the macro inputs are taken to be the sum of micro inputs, the description of the zero-bias $S$-manifold may be rewritten as:

$$G(F_t \cdot \mu_x(t), \theta^M) - F_t \cdot \mu_y(t) = 0. \tag{2.9}$$

Assuming that $G$ is continously differentiable and that $\partial[G(F_t \cdot \mu_x(t), \theta^M) - F_t \cdot \mu_y(t)] \,/\, \partial(F_t \cdot \mu_x(t), F_t \cdot \mu_y(t))$ is non-singular, then by the implicit function theorem $\theta^M$ can, locally, be solved as a function of $(F_t \cdot \mu_x(t), F_t \cdot \mu_y(t))$. Let the solutions to (2.9) be called $\theta^M(F_t \cdot \mu_x(t), F_t \cdot \mu_y(t))$. Then the following result generalizes one of Theil's remarks (1971, p. 570). If the inputs and output aggregates vary over time, a particular solution to (2.9) will vary over time, unless the variations in the output aggregate are proportional to the variations in output resulting from changes in input aggregates. Changes in allocative efficiency or technological changes are incompatible with macro parameters having exact microfoundations that are constant over time. Note that changes in the parameters' aggregates, $\mu_\theta(t)$,—the structural model-change effects (Stoker, 1985)—will appear as variations in the output aggregate $F_t \cdot \mu_y(t)$.

**Proposition 2.** *The exact-aggregation macro parameters* $\theta^M(F_t \cdot \mu_x(t), F_t \cdot \mu_y(t)) \equiv \tilde{\theta}^M(\mu_x, \mu_y, F_t, t)$ *will be constant over time if the inputs and output aggregates*

[19] The following results will provide an additional justification to use time-varying macro parameters. Less vacuous restrictions would be derived from interaction among agents, as in Boyle's law from the theory of gases in physics (see May, 1947).

*are constant over time* $F_t \cdot \mu_x(t) = \bar{\mu}_x$ *and* $F_t \cdot \mu_y(t) = \bar{\mu}_y$ *for all* $t = 1, 2, \ldots, T$, *where* $\bar{\mu}_x$ *and* $\bar{\mu}_y$ *are scalars.*

*Proof:* Follows from the definition of exact-aggregation macro parameters. ◇

**Proposition 3.**

*a) If G is homogeneous of degree* $\lambda_i$ *in the factor inputs* $x_i$ *separately, a necessary and sufficient condition for the exact-aggregation macro parameters* $\theta^M(F_t \cdot \mu_x(t), F_t \cdot \mu_y(t))$ *to be constant over time is that the following proportionality requirements be satisfied:*

$$(2.10) \quad F_t \cdot (\mu_{x_1}(t), \ldots, \mu_{x_N}(t)) = (k_{1t}\bar{\mu}_{x_1}, \ldots, k_{Nt}\bar{\mu}_{x_N}) \quad \text{and} \quad F_t \cdot \mu_y(t) = \kappa_t \bar{\mu}_y,$$

$t = 1, 2, \ldots T$, *where* $\kappa_t = \prod_{i=1}^{N} k_{it}^{\lambda_i}$ *and where* $\mu_x$ *and* $\mu_y$ *are constant over time.*

*b) If G homogeneous of degree* $\lambda$ *in all factor inputs, a sufficient condition for the exact-aggregation macro parameters* $\theta^M(F_t \cdot \mu_x(t), F_t \cdot \mu_y(t))$ *to be constant over time is that the following proportionality requirements be satisfied:*

$$(2.11) \quad F_t \cdot (\mu_{x_1}(t), \ldots, \mu_{x_N}(t)) = (k_t\bar{\mu}_{x_1}, \ldots, k_t\bar{\mu}_{x_N}) \quad \text{and} \quad F_t \cdot \mu_y(t) = k_t^{\lambda}\bar{\mu}_y,$$

$t = 1, 2, \ldots T$, *where* $\bar{\mu}_x$ *and* $\bar{\mu}_y$ *are constant over time. (All factor inputs need to move in the same proportion over time). If G is one-to-one, the requirements are necessary.*

*Proof:* See appendix.

Under constant returns to scale, it suffices that the inputs and output aggregates vary proportionally over time for the macro parameters to be constant over time. This result explains the conclusions of some of Fisher's simulation experiments. Simulating the aggregation of Cobb-Douglas production functions, Fisher (1971)

found that a macro Cobb-Douglas production function, where labor is the variable input, would fit input-output data well if labor's share of total output happens to be roughly constant over time. Since the predictions retained were generated by estimates imposing constant returns, Proposition 3.b applies.

If the time-consistency of the microdefinitions of the macro parameters were not an issue, it would be possible to find solutions to the zero-bias $S$-manifold (2.9) that are constant over time, but whose microdefinitions are time-varying. Let $k(t) = (k_1(t), \ldots, k_N(t))$ and $h(t)$ be some functions of time such that

$$F_t \cdot \mu_x(k(t)) = \bar{\mu}_x \quad \text{and} \quad F_t \cdot \mu_y(h(t)) = \bar{\mu}_y, \quad t = 1, 2, \ldots T, \tag{2.12}$$

at each point in time, then the solution $\theta^M(\bar{\mu}_x, \bar{\mu}_y)$ is constant over time. The time-drifts $k(t)$ and $h(t)$ provide measures of the bias in the parameters in terms of deviations from stable microfoundations.

The type of solution $\theta^M(\bar{\mu}_x, \bar{\mu}_y)$ is likely to be found in empirical analysis. Typically, macroeconometric models are of the form:

$$\bar{y}_t = g(\bar{x}_t, \theta^M) + \epsilon_t, \quad t = 1, 2, \ldots, T \tag{2.13}$$

where $(\bar{y}_1, \ldots, \bar{y}_T, \bar{x}_1, \ldots, \bar{x}_T)$ are observed values of output and inputs aggregates, and $\epsilon_t$ is an error term. The estimates of the macro parameters $\theta^M$ are calculated from the movement of the joint distribution of output and input aggregates over time: $\hat{\theta}^M = \theta(\bar{y}_1, \ldots, \bar{y}_T, \bar{x}_1, \ldots, \bar{x}_T)$. Because $\hat{\theta}^M$ is obtained by curve fitting (2.13), which is the empirical counterpart of the exact aggregation requirement (2.6), the latter will be satisfed almost exactly and, notwithstanding the fitting error, the empirical aggregation bias will appear as unstable microfoundations as in scheme

(2.12). [20]

Currently estimated time-invariant macro parameters will generally not lead to a total output aggregation bias, but their microfoundations will not be stable over time and they will be sensitive to policy changes that are correlated with the explanatory variables and not mean preserving, as will be shown in another paper.

## 3. THE AGGREGATION OF PROFIT FUNCTIONS

### *3.1. The Aggregation of Unrestricted Profit Functions*

In this problem, price-taking behavior on the part of the producer is assumed, but economy-wide equilibrium conditions may or may not prevail. In order for the realizations of the firms' profit characteristics to be independent, it is also assumed that there is no joint production and that each firm's production decisions are taken independently. Different prices for each firm's inputs and output may be assumed. This feature of the model allows the aggregation across industries (across different output prices), by contradistinction with previous exact-aggregation models. [21] Also, this situation could describe an otherwise perfectly competitive economy where royalties, subsidies and indirect taxes introduce price wedges in between firms. Alternatively, the prices could be assumed to differ across firms because of transportation costs as in Geweke (1985).

Regularity conditions are imposed on the production functions in order to make use of duality theory. The assumptions on the production function $g^f$ are: $g^f$ is

[20] Assuming that the error term satisfies normality assumptions, heteroscedastic error terms may capture cross-sectional aggregation biases.

[21] In the empirical analysis of aggregation across an economy, the observed prices would typically take the form of prices indices derived from the ratio of current to constant values, the latter becoming the quantities.

continuous from above, increasing and concave with respect to $x^j$ for all fixed $\theta^j$. Under these conditions, using the augmented production function $g$, the firm's dual profit function is given by:

$$\begin{aligned}\Pi(v^j) \equiv \Pi(p^j, w^j, \theta^j) &\equiv \max_{y^j, x^j} \{p^j y^j - w^j \bullet x^j : y^j \leq g(x^j, \theta^j)\} \\ &\equiv p^j \mathrm{y}(p^j, w^j, \theta^j) - w^j \bullet \mathrm{x}(p^j, w^j, \theta^j),\end{aligned} \tag{3.1}$$

where $v^j = (p^j, w^j, \theta^j) \in \mathbb{R}^{1+N+S}$ are the firm's profit characteristics, $\mathrm{y}(p^j, w^j, \theta^j)$ and $\mathrm{x}_n(p^j, w^j, \theta^j)$, $n = 1, 2, \ldots, N$, are respectively the augmented supply and demand functions.

The profit characteristics $(v^1, \ldots, v^F) \in \mathbb{R}^F \times \mathbb{R}^{1+N+S}$ are interpreted as realizations of a random vector $V \equiv (P, W, \Theta) \in \mathbb{R}^{1+N+S}$. The random vector $V$ is distributed over all firms according to the density $\phi_V(v)$, a joint density of prices and functional parameters. The full-information profit aggregates thus capture not only the distributional properties of the micro relations but also the competitive conditions that prevail in the economy. They can be defined in terms of total profits:

$$\mathit{\Pi} \equiv F \cdot \int_{\Omega_v} \Pi(v)\, \phi_V(v)\, dv \equiv F \cdot \mu_\pi, \tag{3.2}$$

where $\Omega_v \subset \mathbb{R}_+^{1+N} \times \mathbb{R}^S$ is the domain of definition of the profit characteristics and where $\Pi$: $\mathbb{R}^{1+N+S} \rightarrow \mathbb{R}$, so that $\mathit{\Pi} \in \mathbb{R}$. In terms of observed values, the total profit aggregate $\mathit{\Pi}$ corresponds to the economy's profit aggregate used in exact-aggregation models (Gorman, 1968, Blackorby and Schworm, 1984, 1988).

The macro profit function is defined artificially by:

$$\mathit{\Pi}[\psi_v(V)] \equiv \max_{\mathcal{Y}, \mathcal{X}} \{p\mathcal{Y} - w \bullet \mathcal{X} : \mathcal{Y} \leq G(\mathcal{X}, \psi_\theta(\Theta))\}, \tag{3.3}$$

where $\psi_v(V)$ denote aggregates of the profit characteristics and where $\psi_y(Y) = \mathcal{Y}$ and $\psi_x(X) = \mathcal{X}$ represent the aggregate levels of output and inputs and are generally

pre-specified. The real problem is posed by the choice of a macro function $G$, with macro parameters $\psi_\theta(\Theta)$, such that the pre-specified aggregates satisfy (3.3). The total profit-aggregation bias is defined as the difference:

$$B_\pi = F \cdot \int_{\Omega_v} \Pi(v)\, \phi_V(v)\, dv - \Pi[\psi_v(V)]. \tag{3.4}$$

As in output aggregation, the case of an *average-representative* firm, where $G \equiv g$ and $\psi_\theta(\Theta) = \mu_\theta$, is of special interest. Under perfect competition in the sense that $P = \bar{p}$ and $W = \bar{w}$, the profit characteristics become $\psi_v(V) = (\bar{p}, \bar{w}, \mu_\theta)$ and the average profit-aggregation bias is

$$B_{\mu_\pi} = \bar{p}\left(\mathbf{E}_\theta[\mathrm{y}(\bar{p}, \bar{w}, \Theta)] - \mathrm{y}(\bar{p}, \bar{w}, \mu_\theta)\right) - \bar{w} \bullet \left(\mathbf{E}_\theta[\mathrm{x}(\bar{p}, \bar{w}, \Theta)] - \mathrm{x}(\bar{p}, \bar{w}, \mu_\theta)\right). \tag{3.5}$$

**Proposition 4.** *In the context of an average-representative firm, perfect competition may not be sufficient to ensure exact aggregation. The average profit-aggregation bias will be null if and only if expression* (3.5) *vanishes.*

Even under perfect competition, the *average-representative* firm generally will yield a profit-aggregation bias if the supply and demand functions are non-linear in the functional parameters.

In the long-run, however, this bias may disappear as the distribution of firms becomes a degenerate case. The pressure of competition in the longer term may have force the firms to become "redundant" (Makowski, 1980). [22] In these cases, the distribution of the firms' technological parameters may be restricted in such a way as to entail the disappearance of its highers moments. Under perfect competition, the

[22] Agent $i$ is a *redundant* firm if, when $i$ stops being a firm, all agents including $i$ can do as well (Makowski, 1980, p.215).

aggregation problem does not disappear but it is no longer an allocative problem; it reduces to the problem of deriving a macro technology from the micro technologies. This may be facilitated by increasing similarity across firms, which may be, in turn, a long-run result of perfect competition.

In the case where the macro parameters are not pre-specified, the exact-aggregation macro parameters $\theta^M$ will belong to a zero-profit-bias $S$-manifold described by the following equation (introducing time subscripts):

$$\Pi[\mu_p(t), \mu_w(t), \theta^M] - F_t \cdot \mu_\pi(t) = 0, \tag{3.6}$$

where aggregate prices are taken to be average prices over firms and where $S$ is the dimension of the vector $\theta^M$. If the macro profit parameters are required to be constant over time, they will be completely determined over $S$ time periods. If average prices and/or aggregate profits vary over time, the solutions $\theta^M(\mu_p(t), \mu_w(t), F_t \cdot \mu_\pi(t))$ to (3.6) will vary over time, unless the variations in the profit, input and output prices aggregates are all proportional.

**Proposition 5.** *The exact-aggregation parameters $\theta^M(\mu_p(t), \mu_w(t), F_t \cdot \mu_\pi(t))$ of a macro profit function $\Pi(\mu_p(t), \mu_w(t), \theta^M)$ will be constant over time if all prices and profit aggregates vary proportionally over time.*

*Proof:* Since the profit function is homogeneous of degree one in prices, Proposition 3b applies. ◇

The introduction of optimizing behavior and of perfect competition do not eliminate problems of instability in the exact-aggregation macro parameters. In practice, these problems arise as factor and output prices exhibit different inflation rates. Empirical evidence from Canadian data indicates that, in the aggregate from

1961 to 1980, output prices have risen 2.9 fold, capital asset prices 3.1 fold while labor prices increased 4.8 fold. If an approximate aggregation result is judged satisfactory, such divergences may not be important. Empirical analyses are needed to evaluate the order of magnitude of allowable divergences.

### *3.2. The Problem of Capital Aggregation*

Since aggregation problems traditionally were assumed to vanish under conditions of perfect competition, the presence of unmovable capital was long thought to be the main obstacle to the aggregation of the firms' technologies. Thus, the aggregation of capital has been at the center of many exact-aggregation studies (Fisher, 1965, 1969, 1982; Diamond, 1965; Stigum, 1967; Gorman, 1968; Blackorby and Schworm, 1984, 1988). It is reconsidered here by rexamining classic results. [23]

Assume that there are only two types of capital, equipment (e) and structures (s), that perfect competition prevails in the output and labor markets, $P = \bar{p}$ and $W = \bar{w}$, that the firm's augmented restricted profit function is additively separable in the profits attributable to the two types of capital inputs and to the other factors of production, and that the profitability of the two types of capital can be factored into a function of prices and a function of the capital inputs:

(3.7)

$$\Pi(\bar{p}, \bar{w}, \theta^f; e^f, s^f) = c_e(\bar{p}, \bar{w}, \alpha^f) h_e(e^f, a^f) + c_s(\bar{p}, \bar{w}, \beta^f) h_s(s^f, b^f) + c_o(\bar{p}, \bar{w}, \gamma^f),$$

where the components of the profit function are written in augmented form: $c_e(\bar{p}, \bar{w}, \alpha^f)$ and $c_s(\bar{p}, \bar{w}, \beta^f)$ represent the price-dependent (marginal) profitability of equipment and structures, respectively, $h_e(e^f, a^f)$ and $h_s(s^f, b^f)$ represent the firm's rel-

[23] The more recent results (Blackorby and Schworm, 1988) comprise and extend the classic results; they also could be generalized using the principles illustrated here.

ative valuation of the different types of capital, and $c_o(\bar{p}, \bar{w}, \gamma^f)$ denotes the profitability of the other factors of production. If the price-dependent profitability of the different types of capital is the same for all firms, (3.7) corresponds to the functional form prescribed by the Gorman conditions (Gorman, 1968, restated in Blackorby and Schworm, 1984, Theorem 1).

Let A, $\mathcal{A}$, $E$, B, $\mathcal{B}$, $S$ and $\Gamma$ be the random variables whose realizations are $(\alpha^1, \ldots, \alpha^F)$, $(a^1, \ldots, a^F)$, $(e^1, \ldots, e^F)$, $(\beta^1, \ldots, \beta^F)$, $(b^1, \ldots, b^F)$, $(s^1, \ldots, s^F)$ and $(\gamma^1, \ldots, \gamma^F)$. Then, assuming that the macro technology can be described in the same way as the micro technologies are, the variable profit bias is:

$$
\begin{aligned}
B_\pi = {} & F \cdot \Big( \mathbf{E}[c_e(\bar{p}, \bar{w}, \mathrm{A})]\mathbf{E}[h_e(E, \mathcal{A})] + \mathbf{Cov}[c_e(\bar{p}, \bar{w}, \mathrm{A}), h_e(E, \mathcal{A})] \Big) \\
& - c_e(\bar{p}, \bar{w}, \psi_\alpha(\mathrm{A}))h_e(\psi_e(E), \psi_a(\mathcal{A})) + F \cdot \Big( \mathbf{E}[c_s(\bar{p}, \bar{w}, \mathrm{B})]\mathbf{E}[h_s(S, \mathcal{B})] \\
& + \mathbf{Cov}[c_s(\bar{p}, \bar{w}, \mathrm{B}), h_s(S, \mathcal{B})] \Big) - c_s(\bar{p}, \bar{w}, \psi_\beta(\mathrm{B}))h_s(\psi_s(S), \psi_b(\mathcal{B})) \\
& + F \cdot \mathbf{E}[c_o(\bar{p}, \bar{w}, \Gamma)] - c_o(\bar{p}, \bar{w}, \psi_\gamma(\Gamma)),
\end{aligned}
\tag{3.8}
$$

where $\mathbf{Cov}(\cdot, \cdot)$ is the covariance operator.

The Gorman conditions first comprise a set of distributional assumptions, $\mathrm{A} = \bar{\alpha}$ and $\mathrm{B} = \bar{\beta}$, which make the covariance terms in (3.8) vanish. They also ensure, if the $c_r(\cdot)$, $r = e, s$, functions are linear in the parameters, that the expected price-dependent profitabilities equal that of the *average-representative* firm

$$
\begin{aligned}
& \mathbf{E}[c_r(\bar{p}, \bar{w}, \Xi)] = c_r(\bar{p}, \bar{w}, \bar{\xi}) = c_r(\bar{p}, \bar{w}, \psi_\xi(\Xi)) \\
& \text{if} \quad \Xi = \bar{\xi} = \mathbf{E}(\Xi) = \psi_\xi(\Xi) \qquad \xi = \alpha, \beta \quad r = e, s.
\end{aligned}
\tag{3.9}
$$

Assumptions that are less stringent distributionally would require that the covariance between the $c_r(\cdot)$, $r = e, s$, functions and the $h_r(\cdot)$, $r = e, s$, functions be null or, more generally, that the sum of these covariances vanish. [24] The need

[24] In terms of the Gorman Polar form (Gorman, 1953), $x^f = \alpha^f(p) + \beta(p)\, u^f$

for some sort of "independence" between the $c_r(\cdot)$, $r = e, s$, functions and the $h_r(\cdot)$, $r = e, s$, functions has been recognized previously by Lau (1982) who required that analogous functions be "linearly independent" (Lau, 1982, Theorem 2, p. 213). The above zero-covariances-sum condition is however a weaker assumption, especially if an approximate result is judged satisfactory in empirical analysis.

In effect, the stochastic independence of the price-dependent profitability of the capital inputs and of the firms' valuation of these different types of capital seems unlikely as a result of the optimization process. However, the annihilation of the covariances could follow from sufficient diversity in the economy. If a firm (1) relies on a high level of structures relative to its level of equipment, because of the technological processes involved (e.g. mining), it may attribute a high valuation to its structures ($h_s^1$ = high), despite a profitability that could be comparatively low ($c_s^1$ = low). Whereas a firm (2) that uses a relatively low level of structure (e.g. chemical industries) may value it correspondingly low ($h_s^2$ = low) with an equally low profitability ($c_s^2$ = low). In such circumstances, the covariance between the $h_s(\cdot)$ and the $c_s(\cdot)$ functions could be small. [25] That is, the technologically constrained valuations could be independent of the relative marginal profitabilities, if firms utilize sufficiently diverse technologies. The extent to which firms can differ technologically within the same industry may be questionable. But this result widens consistent aggregation possibilities across an economy.

A second set of conditions (Gorman, 1968) construct capital aggregates on the

---

($f = 1, 2, \ldots F$), it would suffice that $\mathbf{Cov}(\mathrm{B}, U) = 0$ instead of $\mathrm{B} \equiv \bar{\beta}$ for all $f$, where B and $U$ are the random variables whose realizations are $(\beta^1, \ldots, \beta^F)$ and $(u^1, \ldots, u^F)$.

[25] Alternatively, it could be counterbalanced by the covariance between the $h_e(\cdot)$ and the $c_e(\cdot)$ functions.

basis of the firms' valuation of the different types of capital (also called capital subaggregates): $\psi_{h_e}(h_e(E,A)) \equiv F \cdot \mathbf{E}[h_e(E,A)]$ , $\psi_{h_s}(h_s(S,B)) \equiv F \cdot \mathbf{E}[h_s(S,B)]$, which are equal to $\sum_{f=1}^{F} h_e(e^f, a^f)$ and $\sum_{f=1}^{F} h_s(s^f, b^f)$ in terms of observed values, respectively. These types of aggregates have the important disadvantage of requiring perfect knowledge of the micro relations to be computed and possess some empirical value only when additional distributional information is available. The case where $h_e(e^f, a^f) \equiv e^f$ and $h_s(s^f, b^f) \equiv s^f$, used in the Gorman Polar Form, is of course an exception to this complication. Studying consumer behavior, Jorgenson, Lau and Stoker (1982) have called similar aggregates, index functions, and have correctly interpreted them as statistics describing the population. The present stochastic approach has the advantage of suggesting which precise "statistics" are needed to aggregate exactly specific micro relations.

An alternative solution, using the usual first moments as aggregates, $\psi_e(E) = F \cdot \mathbf{E}(E)$ and $\psi_s(S) = F \cdot \mathbf{E}(S)$, would require that the $h_r(\cdot)$, $r = e, s$, functions be affine in $E$ and $S$ with $\mathbf{Cov}(E,A) = \mathbf{Cov}(E,\mathrm{A}) = \mathbf{Cov}(A,\mathrm{A}) = 0$ and $\mathbf{Cov}(S,B) = \mathbf{Cov}(S,\mathrm{B}) = \mathbf{Cov}(B,\mathrm{B}) = 0$. More importantly, in a stochastic framework, it is no longer necessary that the profit function be additively separable in the two types of capital inputs. A mixed term, for example $h_{e,s}(e^f, s^f, \delta^f) \equiv \delta^f e^f s^f$, could be added to (3.7) as long as $\mathbf{Cov}(E,S) = \mathbf{Cov}(E,\Delta) = \mathbf{Cov}(S,\Delta) = 0$. For each individual firm, the relative quantities of equipment and structures are likely to be dependent as a result of the optimization process. However, if the economy is made up of diverse enough technologies, the covariance between the two types of capital could be small or could be annihilated by counterbalancing moments.

Since the proposed alternative solutions to Gorman's solution are not a subset

of the latter, the Gorman conditions cannot be seen as necessary in a stochastic framework, and they are sufficient only if capital aggregates that depend on micro characteristics are available. Exact aggregation now seems to propose a clear choice between either imposing simple affine forms with a high degree of similarity across agents or gathering the information (in terms of higher and central moments) that makes it possible to validate or complete the aggregation of less restrictive forms.

## 4. EMPIRICAL EVIDENCE FROM SECTORAL DATA

Using sectoral data from the Canadian input-output tables, empirical investigations were conducted to assess the plausibility of the theoretical results. The input-output tables are given for 37 sectors of the Canadian Economy, a level of aggregation comparable to the U.S. SIC two-digit classification system. The data is annual and covers the period 1961 to 1980. Different deflators in terms of producers' prices were used for each sector, so that price data for each sector could be computed. The constant dollars (real) values are used as quantities.

Considering the first problem of the average representative, the empirical analyses serve *i)* to find whether the conditions for the existence of an *average-representative* firm could possibly be met by "real" data, *ii)* to assess whether the divergences brought by the use of non-linear functional forms are important. [26] Secondly, the question of the instability of the exact-aggregation macro parameters is examined using a linear production function.

A simple production function of two inputs, capital and labor, is used to illustrate the bias implications of alternative functional forms. The data includes

[26] The term "firm" is used loosely to mean a unit of production; a sector is a semi-aggregated "unit" of production.

five types of capital goods: raw and finished inventories, material & equipment, structures and land. Intrasectoral capital aggregates of the five capital inputs were formed using Divisia quantity indices under various specifications of producers' expectations of future capital prices in the capital rental prices formula (case a: static expectations, case b: smoothed expectations). [27] Because it was not possible to use the internal rate of return in the calculation of capital rental prices, a common external rate of return was utilized with disastrous consequences for the aggregation biases. [28] These were also computed with a capital aggregate based on asset prices for comparison (case c).

### *4.1. Average-Representative Firm's Bias*

A first series of empirical evaluations serves to find: how far is the output prediction of the *average-representative* firm from the "true" output aggregate? A linear production function of capital and labor was estimated for each firm using the following model:

$$y_t^f = \alpha_1^f + \alpha_k^f k_t^f + \alpha_l^f l_t^f + \epsilon_t^f, \quad t = 1,\ldots,T, \tag{4.1}$$

where $\epsilon_t^f$ is a normally distributed random variable (over time) with $\mathbf{E}(\epsilon^f) = 0$ and $\mathbf{E}(\epsilon^{f\prime}\epsilon^f) = \sigma_f^2 I$. These assumptions about the disturbances are maintained for all sectors, data sets and functional forms.

Let upper case letters denote the stochastic (across sectors) variables whose realizations are denoted by lower case letters. The expectation (across sectors) of

[27] The formula used is the one proposed in Diewert (1980, p.471). The case of perfect expectations lead to highly erratic and negative rental prices (as argued by Denison, 1969,p.9). These were not deemed reasonable and consequently left out of further analysis. Details on the formation of the capital prices are available in Fortin (1988).

[28] The rate of return used was the McLeod, Young and Weir's 10 industrial bonds yield average, the counterpart to the American Moody bond rate.

estimates of (4.1) may be written equivalently as

$$(4.2) \qquad \mathbf{E}(Y_t) = \mathbf{E}(\hat{\mathrm{A}}_1) + \mathbf{E}(\hat{\mathrm{A}}_k K_t) + \mathbf{E}(\hat{\mathrm{A}}_l L_t) + \mathbf{E}(\epsilon_t) = \mathbf{E}(\hat{Y}_t) + \mathbf{E}(\epsilon_t),$$

and the estimated production function of an *average-representative* firm is

$$(4.3) \qquad \hat{Y}_t^r = \mathbf{E}(\hat{\mathrm{A}}_1) + \mathbf{E}(\hat{\mathrm{A}}_k)\mathbf{E}(K_t) + \mathbf{E}(\hat{\mathrm{A}}_l)\mathbf{E}(L_t).$$

The resulting output aggregation bias can be estimated equivalently using

$$(4.4) \qquad B_{\mu_b}(t) = \mathbf{Cov}(\hat{\mathrm{A}}_k, K_t) + \mathbf{Cov}(\hat{\mathrm{A}}_l, L_t) = \mathbf{E}(\hat{Y}_t) - \hat{Y}_t^r.$$

[INSERT TABLE I]

In another set of estimations, a quadratic production function of capital and labor was fitted to the data using the model:

$$(4.5) \quad y_t^f = \alpha_1^f + \alpha_k^f k_t^f + \alpha_l^f l_t^f + \alpha_{kk}^f (k_t^f)^2 + \alpha_{kl}^f k_t^f l_t^f + \alpha_{ll}^f (l_t^f)^2 + \epsilon_t^f, \quad t = 1, \ldots, T,$$

where $\epsilon_t^f$ is a disturbance term. [29] The average (across sectors) output aggregate is given by

$$(4.6) \qquad \begin{aligned} \mathbf{E}(Y_t) = {} & \mathbf{E}(\hat{\mathrm{A}}_1) + \mathbf{E}(\hat{\mathrm{A}}_k K_t) + \mathbf{E}(\hat{\mathrm{A}}_l L_t) + \mathbf{E}(\hat{\mathrm{A}}_{kk} K_t^2) + \mathbf{E}(\hat{\mathrm{A}}_{kl} K_t L_t) \\ & + \mathbf{E}(\hat{\mathrm{A}}_{ll} L_t^2) + \mathbf{E}(\epsilon_t)\mathbf{E}(\hat{Y}_t) + \mathbf{E}(\epsilon_t), \end{aligned}$$

and the estimated production function of an *average-representative* is

$$(4.7) \qquad \begin{aligned} \hat{Y}_t^r = {} & \mathbf{E}(\hat{\mathrm{A}}_1) + \mathbf{E}(\hat{\mathrm{A}}_k)\mathbf{E}(K_t) + \mathbf{E}(\hat{\mathrm{A}}_l)\mathbf{E}(L_t) + \mathbf{E}(\hat{\mathrm{A}}_{kk})\mathbf{E}(K_t^2) + \mathbf{E}(\hat{\mathrm{A}}_{kl})\mathbf{E}(K_t)\mathbf{E}(L_t) \\ & + \mathbf{E}(\hat{\mathrm{A}}_{ll})\mathbf{E}(L_t^2). \end{aligned}$$

[29] Estimates of these parameters showed that in most cases, the production function was concave at the observed values. Exceptions were in case a), the tobacco, broadcasting and telephone sectors; in case b), the agricultural, tobacco and petroleum sectors; in case c), the agricultural, water and motor transportation sectors.

The output aggregation bias of the *average-representative* firm can be estimated using: $B_{\mu_k}(t) = \mathbf{E}(\hat{Y}_t) - \hat{Y}_t^r$ or the equivalent expression in terms of variances-covariances.

The bias' results are given in Table I. In the linear case, there exists a representative firm (case 1a), whose technology is derived from that of non-identical firms, that predicts the aggregate level of output with reasonable accuracy (in the order of the fitting error). The untampered data provided sufficient technological diversity, but when a common rate of return was used to compute the capital aggregate (case b and c), that diversity was lost. In the quadratic case, the order of magnitude of the *average-representative* firm 's bias is no longer acceptable. [30]

*4.2. Exact-Aggregation Macro Parameters with Stable Microfoundations*

The time-inconsistency of exact-aggregation macro parameters is investigated next. The macro parameters are estimated using a macro relation and then their correspondence to the micro parameters and variables is established.

Let the macro production function be a linear function of input and output aggregates, defined in terms of averages

$$\mathbf{E}(Y_t) = \alpha_1^M + \alpha_k^M \mathbf{E}(K_t) + \alpha_l^M \mathbf{E}(L_t) + \epsilon_t^M. \tag{4.8}$$

The estimated macro production function then is given by

$$\begin{aligned} &\mathbf{E}(\hat{Y}_t) = \hat{\alpha}_1^M + \hat{\alpha}_k^M \mathbf{E}(K_t) + \hat{\alpha}_l^M \mathbf{E}(L_t) \\ \text{or} \quad &\mathbf{E}(\hat{\mathrm{A}}_1) + \mathbf{E}(\hat{\mathrm{A}}_k K_t) + \mathbf{E}(\hat{\mathrm{A}}_l L_t) = \hat{\alpha}_1^M + \hat{\alpha}_k^M \mathbf{E}(K_t) + \hat{\alpha}_l^M \mathbf{E}(L_t). \end{aligned} \tag{4.9}$$

[30] A separable quadratic production function was also fitted to the data. The resulting bias estimates were similar to the non-separable case. If the covariance terms in the population of characteristics are important, the imposition of separable functional form will not eliminate them; it will simply displace them.

Using equation (4.9), the exact-aggregation macro parameters can be described by the following family of microfoundations:

$$
\begin{aligned}
\hat{\alpha}_1^M &= \lambda_1(t)\mathbf{E}(\hat{\mathrm{A}}_1) + \lambda_2(t)\mathbf{E}(\hat{\mathrm{A}}_k K_t) + \lambda_3(t)\mathbf{E}(\hat{\mathrm{A}}_l L_t), \\
\hat{\alpha}_k^M &= \frac{\gamma_1(t)\mathbf{E}(\hat{\mathrm{A}}_1) + \gamma_2(t)\mathbf{E}(\hat{\mathrm{A}}_k K_t) + \gamma_3(t)\mathbf{E}(\hat{\mathrm{A}}_l L_t)}{\mathbf{E}(K_t)}, \\
\hat{\alpha}_l^M &= \frac{\delta_1(t)\mathbf{E}(\hat{\mathrm{A}}_1) + \delta_2(t)\mathbf{E}(\hat{\mathrm{A}}_k K_t) + \delta_3(t)\mathbf{E}(\hat{\mathrm{A}}_l L_t)}{\mathbf{E}(L_t)},
\end{aligned}
\tag{4.10}
$$

where the time-drift coefficients $\lambda_i(t) + \gamma_i(t) + \delta_i(t) = 1$, $i = 1, 2, 3$, for all $t = 1, 2, \ldots, T$. Using three time periods at a time, that is assuming that the microdefinitions of the macro parameters are identical over three year periods, equations (4.10) can be solved exactly for $\lambda$, $\gamma$ and $\delta$, provided that the corresponding solution matrix is non-singular.

[INSERT TABLE II ]

The effects of distributional changes (from capital to labor) due to the 75-76 oil-shock on exact-aggregation macro parameters are illustrated in Table II. The (75-76-77) time-drifts coefficients have been inserted in (4.10) and macro parameters with the corresponding microdefinitions were computed. The original macro parameters ($\hat{\alpha}_k^M$ and $\hat{\alpha}_l^M$) are found under the corresponding headings in the years 75-76-77. In case a), the macro parameters show important drifts outside the confidence intervals of the original parameters. Because of expectations smoothing, this anomaly does not appear in case c).

When time-drift coefficients based on periods of relative economic stability were used, the exact-aggregation macro parameters with consistent microdefinitions did not fluctuate outside of their confidence intervals. The analysis shows that generally the potential biases in exact-aggregation macro parameters due to distributional changes may be relatively small.

## 5. CONCLUDING REMARKS

The stochastic framework presented here provides a simple device to analyze the importance of aggregation problems. It could be used by macro modelers to study the effects of aggregation distortions on macroeconometric policy evaluation, by empirical analysts to evaluate the seriousness of potential aggregation biases.

The model shows that traditional aggregation restrictions may be relaxed with the help of distributional assumptions. Modeling aggregation problems in an environment where the distribution over agents matters, enlarges the set of exact (and approximate) aggregation possibilities. An interesting set of circumstances that may minimize aggregation biases is provided by economic diversity. It validates the intuitive notion that a "true" representative may exist when the economy is made up of agents that are either very similar or so different that individual differences cancel each other out, as proposed by Cournot (1838) to justify the regularity of market demand functions. [31] Empirical experiments to specify the nature and extent of the required diversity are underway.

Another result takes the form of a general exact-aggregation manifold relating micro characteristics to macro parameters. It indicates that estimated macro parameters generally will not possess stable microfoundations. Unless the population of explanatory variables changes over time in proportion with the explained variable, exact aggregation implies that the macro parameters will be unstable if their microdefinitions remain the same over time. Empirical analyses confirm that, in

[31] "But the wider the market extends, and the more the combinations of needs, of fortunes, or even of caprices, are varied among consumers, the closer the function $F(p)$ [the demand function] will come to varying with $p$ in continuous manner." (Cournot, 1838/1929, p.50).

the presence of shocks (here the 75-76 oil shock), the distortions brought about by distributional effects alone, are sufficiently significant to state that macro models estimated prior to important disturbances will not predict accurately the aftermath.

In a stochastic framework, the aggregation of optimized relations is not intrinsically different from the aggregation of non-optimized relations. If the ultimate aim is the construction of a microfounded macro function, perfect competition alone does not guarantee that the macro parameters can be derived from the corresponding micro parameters. In particular, it does not necessarily imply that an *average-representative* will be an exact-representative. Furthermore, the time-consistency of exact-aggregation macro parameters may be questioned even under perfect competition, unless the variations in explained and explanatory prices are all proportional. However, if the long run competitive pressures force firms to become alike, then the increased similarity will ease the derivation of a representative technology.

The theoretical reasons why the "macro" description of aggregate economic relations can lead to aggregation biases are simple. First, the stochastic interdependence of the agents' characteristics may imply the existence of covariance terms, even in linear relationships, that are not taken into account by usual aggregate data. In particular, in macroeconometric modeling, the omission of covariance terms between functional parameters and explanatory variables may result in aggregation biases with respect to policy changes. If an exact-aggregation representative agent approach is adopted, policy changes may be assumed to lead to shifts in the macro parameters, unless the policy changes are stochastically independent from the functional parameters and are mean-preserving. Secondly, if the underlying relationships are non-linear, their aggregation will entail the existence of second and higher mo-

ments of the characteristics, which again are not described by aggregate data. [32] The usual aggregates, the totals and averages, are informationally inadequate in those circumstances. The same remark applies to more sophisticated aggregates (such as Divisia indexes) that, nevertheless, do not capture the essential features of the micro relations.

The question of whether the consequences of the inconsistencies of representative agent macro models, are important or can be neglected is an empirical question. The illustrative results presented here may not be widely transferable, however they signal no cause of alarm but serve a word of caution. Much empirical research is needed to assess the importance of aggregation errors.

*(March 15, 1989)*

## APPENDIX

Proof of Proposition 1. (Series' form of the average output aggregation bias)

*Proof:* Express the production function $g$ evaluated at $z$ in terms of its Taylor series expansion

$$y = g(z) = g(\bar{z}) + (z - \bar{z})'\nabla\bar{g} + \tfrac{1}{2}(z - \bar{z})'\bar{H}(z - \bar{z}) + \cdots, \tag{A.1}$$

where $\nabla\bar{g}$ is a vector of first order derivatives of $g(z)$ and $\bar{H}$ is the Hessian matrix, both evaluated at $\bar{z}$, and assume that the matrix of third order derivatives is bounded.

[32] Problems arising from the absence of variance data have been pointed out previously by various authors.

Substituting (A.1) into (2.2) gives

(A.2)

$$\mu_y = \int_{\Omega_z} [g(\tilde{z}) + (z-\tilde{z})'\nabla\tilde{g} + \tfrac{1}{2}(z-\tilde{z})'\tilde{H}(z-\tilde{z}) + \cdots]\phi_Z(z)\,dz$$

$$= g(\tilde{z}) + \nabla\tilde{g}\int_{\Omega_z}(z-\tilde{z})\phi_Z(z)\,dz + \tfrac{1}{2}\mathrm{tr}\tilde{H}\int_{\Omega_z}(z-\tilde{z})'(z-\tilde{z})\phi_Z(z)\,dz + \cdots,$$

since the trace operator is associative.

The average output aggregation bias is then given by

(A.3) $$B_{\mu_y}(\tilde{z}) = \mu_y - g(\tilde{z}) = (\mu_z - \tilde{z})'\nabla\tilde{g} + \tfrac{1}{2}\mathrm{tr}\tilde{H}\left[\mathrm{E}(Z^2) - 2\tilde{z}'\mu_z + \tilde{z}'\tilde{z}\right] + \cdots$$

Clearly at $\tilde{z} = \psi_z(Z) = \mu_z$, the first order term vanishes, the bias is then identically equal to the sum of higher order terms. ◇

Proof of Proposition 2. (Necessary and sufficient conditions for the time constancy of exact-aggregation macro parameters)

*Proof: a)* For the exact-aggregation macro parameters to be constant over time, the zero-bias $S$-manifold (2.9) has to lead to the same solutions over time. Assuming two time periods $t = 0, 1$, let $\mu_x(0)$ and $\mu_y(0)$ be the inputs and output aggregates at time 0 and express the inputs and output aggregates of period 1 as a proportion of the aggregates of period 0

(A.4) $$F_1 \cdot (\mu_{x_1}(1), \ldots, \mu_{x_N}(1)) = F_0 \cdot (k_{11}\mu_{x_1}(0), \ldots, k_{N1}\mu_{x_N}(0))$$

(A.5) $$\text{and} \quad F_1 \cdot \mu_y(1) = h_1\, F_0 \cdot \mu_y(0).$$

In period 1, the zero-bias $S$-manifold is now described by:

(A.6) $$G\left(k_{11}F_0 \cdot \mu_{x_1}(0), \ldots, k_{N1}\,F_0 \cdot \mu_{x_N}(0), \theta^M\right) - h_1\; F_0 \cdot \mu_y(0) = 0.$$

Since $G$ is homogeneous of degree $\lambda_i$ in each $x_i$, equation (A.6) can be rewritten as

(A.7) $$\kappa_1 G\left(F_0 \cdot \mu_x(0), \theta^M\right) - h_1\; F_0 \cdot \mu_y(0) = 0,$$

where $\kappa_1 = \prod_{i=1}^{N} k_{i1}^{\lambda_i}$. In period 0, the zero-bias $S$-manifold is described by

$$G\left(F_0 \cdot \mu_x(0), \theta^M\right) - F_0 \cdot \mu_y(0) = 0. \tag{A.8}$$

Clearly, the period 1 $S$-manifold will be identical to the period 0 $S$-manifold if and only if $h_1 = \kappa_1$.

*b)* Sufficiency is proven using the same demonstration as in *a)* with the corresponding proportions (2.11) for inputs and output aggregates between period 0 and period 1. To prove necessity, assume that $\theta^M$ are solutions to the period 0 S-manifold. Obtain the expression for the output aggregate in period 0 in terms of the macro function from (A.8), substitute it into the expression for the output aggregate in period 1 as a proportion of the output aggregate in period 0 (A.5) and obtain

$$F_1 \cdot \mu_y(1) = h_1 G\left(F_0 \cdot \mu_x(0), \theta^M\right). \tag{A.9}$$

If $\theta^M$ is also a solution to period 1 S-manifold, then using (A.9),

$$G\left(F_1 \cdot \mu_x(1), \theta^M\right) - h_1 G\left(F_0 \cdot \mu_x(0), \theta^M\right) = 0. \tag{A.10}$$

Since $G$ is homogeneous of degree $\lambda$ in the factor inputs, equation (A.10) can be rewritten as

$$G\left(F_1 \cdot \mu_x(1), \theta^M\right) - G\left(h_1^{1/\lambda} F_0 \cdot \mu_x(0), \theta^M\right) = 0. \tag{A.11}$$

Since $G$ is one-to-one, equation (A.11) implies $F_1 \cdot \mu_x(1) = h_1^{1/\lambda} F_0 \cdot \mu_x(0)$. $\diamond$

## REFERENCES

Allen, R.G.D. (1959): *Mathematical Economics*, London: MacMillan.

Biser, D.S. and D.E. Sichel (1988): "Irreversible Investment: An Empirical Investigation," Paper presented at the Econometric Society Winter Meetings.

Blackorby, C., and W. Schworm (1984): "The Structure of Economies with Aggregate Measures of Capital: A Complete Characterization," *Review of Economic Studies*, 51, 633-650.

——— (1988): "The Existence of Input and Output Aggregates in Aggregate Production Functions," *Econometrica*, 56, 613-643.

Casarosa, C. (1981): "The Microfoundations of Keynes's Aggregate Supply and Expected Demand Analysis," *The Economic Journal*, 91, 188-194.

Chetty, V.K. and J.J. Heckman (1986): "A Dynamic Model of Aggregate Output Supply, Factor Demand and Entry and Exit for a Competitive Industry with Heterogeneous Plants," *Journal of Econometrics*, 33, 237-262.

Cournot, A. (1929): *Researches into the Mathematical Principles of the Theory of Wealth*, (Recherches sur les principes mathématiques de la théorie des richesses, 1838), transl. by N.T. Bacon, New York: MacMillan Co.

Cox, J.C., J.E. Ingersoll, Jr., and S.A. Ross (1985): "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, 363-384.

Denison, E.F. (1969): "Some Major Issues in Productivity Analysis: An Examination of Estimates by Jorgenson and Griliches," *Survey of Current Business*, 49, 1-27.

Diamond, P.A. (1965): "Technical Change and the Measurement of Capital and Output," *Review of Economic Studies*, 32, 289-298.

Diewert, W.E. (1980): "Aggregation Problems in the Measurement of Capital," in *The Measurement of Capital*, ed. by D. Usher, Chicago: University of Chicago Press, 433-538.

Fisher, F.M. (1965): "Embodied Technical Change and the Existence of an Aggregate Capital Stock," *Review of Economic Studies*, 32, 263-288.

——— (1969): "The Existence of Aggregate Production Functions," Econometrica, 37,553-577.

——— (1971): "Aggregate Production Functions and the Explanation of Wages: A Simulation Experiment," *Review of Economics and Statistics*, 53, 305-325.

——— (1982): "Aggregate Production Functions Revisited: The Mobility of Capital and the Rigidity of Thought," *Review of Economic Studies*, 49, 615-626.

Fortin, N.M. (1988): "Evaluating the Aggregation Biases in a Production Economy: A Stochastic Approach", University of British Columbia Ph.D. Dissertation, Vancouver, B.C.

Friedman, M. (1976): "Comments on Tobin and Buiter," in *Monetarism*, ed. by J.L. Stein, Amsterdam: North-Holland, 310-317.

Geweke, J. (1985): "Macroeconometric Modeling and the Theory of the Representative Agent," *American Economic Review, Papers and Proceedings*, 75, 206-210.

Gorman, W.M. (1953): "Community Preference Fields," *Econometrica*, 21, 63-80.

——— (1968): "Measuring the Quantities of Fixed Factors," in *Value, Capital and Growth: Papers in Honour of Sir John Hicks*, ed. by J.N. Wolfe, Edinburgh: Edinburgh University Press, 141-172.

Hildenbrand, W. (1981): "Short-Run Production Functions Based on Microdata," *Econometrica*, 49, 1095-1125.

Houthakker, H.S. (1955): "The Pareto Distribution and the Cobb-Douglas Production Function in Activity Analysis," *Review of Economic Studies*, 23, 27-31.

Jensen, J.L.W.V. (1906): "Sur les functions convexes et les inégalités entre les valeurs moyennes," *Acta Mathematica*, 30, 175-193.

Jevons, W.S. (1879): *The Theory of Political Economy*, London: MacMillan.

Johansen, L. (1972): *Production Functions — An Integration of Micro and Macro, Short Run and Long Run Aspects*, Amsterdam: North-Holland.

Jorgenson, D.W, L.J. Lau and T.M. Stoker (1982): "The Transcendental Logarithmic Model of Aggregate Consumer Behavior," in *Advances in Econometrics*, ed. by R.L. Basmann and G.F. Rhodes, Jr., Greenwich, Conn.: JAI Press Inc., vol. 1, 97-238.

Keller, W.J. (1980): "Aggregation of Economic Relations: A Note," *De Economist*, 128, 558-562.

Klein, L.R. (1946): "Remarks on the Theory of Aggregation," *Econometrica*, 14, 303-312.

Kydland, F.E. and E.C. Prescott (1982): "Time to Build and Aggregate Fluctuations," *Econometrica*, 50, 1345-1370.

Lau, L.J. (1982): "A Note on the Fundamental Theorem of Exact Aggregation," *Economic Letters*, 9, 119-126.

Lewbel, A. (1986a): "Exact Aggregation and a Representative Agent: Towards a Reconciliation," Brandeis University Discussion Paper.

——— (1986b): "Log-log and Log-linear Macro Models," Brandeis University Working Paper # 126-1986.

Lucas, R.E. Jr. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1426-1446.

Lucas, R.E. Jr. and L. Rapping (1969): "Real Wages, Employment and Inflation," *Journal of Political Economy*, 77, 721-754.

Makowski, L. (1980): "A Characterization of Perfectly Competitive Economies with Production," *Journal of Economic Theory*, 22, 208-221.

Marshall, A. (1929): *Principles of Economics*, 8th ed., MacMillan: London.

May, K. (1947): "Technological Change and Aggregation," *Econometrica*, 15, 51-63.

Menger, Carl (1950): *Principles of Economics*, (Grundsätze, 1871), transl. and ed. by J. Dingwall and B.F. Hoselitz, Glencoe, Ill.: The Free Press.

Nataf, A. (1948): "Sur la possibilité de construction de certains macromodèles," *Econometrica*, 16, 232-244.

Ry, K., "Technological Change and Aggregation," *Econometrica*, 15, 1947, 51-63.

Richmond, J. (1976): "Aggregation and Identification", *International Economic Review*, 17, 47-56.

Sargent, T.J. (1978): "Estimation of Dynamic Labor Demand Schedules under Rational Expectations," *Journal of Political Economy*, 86, 1009-1044.

Sato, K. (1975): *Production Functions and Aggregation*, Amsterdam: North-Holland.

Stigum, B.P. (1967): "On Certain Problems in Aggregation," *International Economic Review* , 8, 349-367.

Stoker, T.M. (1984): "Completeness, Distribution Restrictions, and the Form of Aggregate Functions," *Econometrica*, 52, 887-907.

——— (1985): "Aggregation, Structural Change and Cross-Section Estimation," *Journal of the American Statistical Association*, 80, 720-729.

——— (1986): "Aggregation, Efficiency and Cross-Section Regression," *Econometrica*, 54, 171-188.

Theil, H. (1954): *Linear Aggregation of Economic Relations*, Amsterdam: North-Holland.

——— (1971): *Principles of Econometrics*, New York: John Wiley & Sons.

Van Daal, J. (1980): "Money Illusion and Aggregation Bias," *De Economist*, 128, 86-93.

Van Daal, J. and A.H.Q.M. Merkies (1984): *Aggregation in Economic Research: from Individual to Macro Relations*, Dordrecht: D. Reidel.

TABLE I

Average Representative Firm's Bias (t=1970)

| | $E(Y_t) - E(\hat{Y}_t)$[a] | $1 - \frac{E(\hat{Y}_t)}{E(Y_t)}$ | $E(\hat{Y}_t) - \hat{Y}_t^r$[a] | $1 - \frac{\hat{Y}_t^r}{E(\hat{Y}_t)}$ |
|---|---|---|---|---|
| 1. Linear production function using capital inputs based on | | | | |
| a) Asset Prices | 52615.5781 | 0.020 | 145477.6250 | 0.056 |
| b) Rental Prices with Static Expectations | 84743.3750 | 0.032 | -435280.1250 | -0.169 |
| c) Rental Prices with Smoothed Expectations | -37215.0117 | -0.014 | -1213248.0000 | -0.450 |
| 2. Quadratic production function using capital inputs based on | | | | |
| a) Asset Prices | -5067.9570 | -0.002 | -1961539. | -0.737 |
| b) Rental Prices with Static Expectations | -26192.1484 | -0.010 | -14696180. | -5.479 |
| c) Rental Prices with Smoothed Expectations | -125342.3750 | -0.047 | 26395152. | 9.490 |

[a] (000 $CAN)

## TABLE II

### Macroparameters with Stable Microfoundations (75-76-77 Time Drift Coefficients)

| | a) Asset Prices | | | b) Rental Prices with Smoothed Expectations | | |
|---|---|---|---|---|---|---|
| Years | $\alpha_1$ | $\alpha_k$ | $\alpha_l$ | $\alpha_1$ | $\alpha_k$ | $\alpha_l$ |
| 1961 | -7845641. | 0.024368 | 15.5215 | -4216089. | 0.003750 | 9.0781 |
| 1962 | -7520342. | 0.022751 | 14.5652 | -4244187. | 0.002797 | 9.0510 |
| 1963 | -7249876. | 0.021354 | 13.9159 | -4263961. | 0.002047 | 9.0708 |
| 1964 | -6787346. | 0.019386 | 12.7904 | -4297307. | 0.000749 | 9.0551 |
| 1965 | -6146339. | 0.016863 | 11.3678 | -4356772. | -0.000887 | 9.0599 |
| 1966 | -5681666. | 0.015054 | 10.4967 | -4382079. | -0.002571 | 9.1042 |
| 1967 | -5638895. | 0.014544 | 10.4124 | -4350887. | -0.003518 | 9.0753 |
| 1968 | -5624248. | 0.014333 | 10.4067 | -4295981. | -0.004422 | 9.1003 |
| 1969 | -5400527. | 0.013491 | 10.0434 | -4253501. | -0.005684 | 9.0101 |
| 1970 | -5512689. | 0.013572 | 10.3179 | -4185565. | -0.006121 | 8.9979 |
| 1971 | -5444699. | 0.013202 | 10.1858 | -4202537. | -0.006380 | 8.9916 |
| 1972 | -5043435. | 0.012269 | 9.4724 | -4232387. | -0.006928 | 8.9591 |
| 1973 | -4442130. | 0.011160 | 8.4646 | -4318368. | -0.007821 | 8.9435 |
| 1974 | -4024298. | 0.010469 | 7.8063 | -4297503. | -0.008731 | 8.8315 |
| 1975 | -4260461. | 0.010762 | 8.1743 | -4241891. | -0.008373 | 8.6871 |
| 1976 | -4181683. | 0.010762 | 8.1743 | -4230969. | -0.008373 | 8.6871 |
| 1977 | -4161088. | 0.010762 | 8.1743 | -4271646. | -0.008373 | 8.6871 |
| 1978 | -3789602. | 0.010449 | 7.6578 | -4184645. | -0.008809 | 8.5719 |
| 1979 | -3369891. | 0.010131 | 7.1122 | -3903623. | -0.009730 | 8.3680 |
| 1980 | -3313239. | 0.010203 | 7.1604 | -3448230. | -0.010143 | 8.1210 |